(Non-)decay for the massless Vlasov equation on subextremal and extremal Reissner–Nordström

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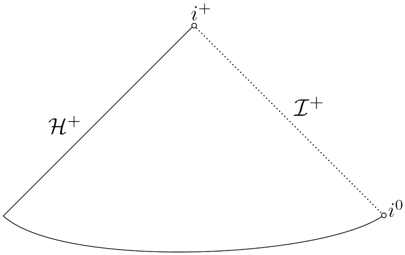

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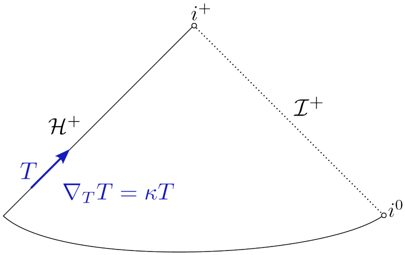

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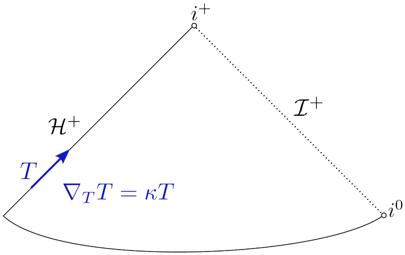

Recently tremendous progress on stability of subextremal black holes, e.g.

Dafermos–Holzegel–Rodnianski–Taylor ('22), Giorgi–Klainerman–Szeftel ('22), Angelopoulos–Aretakis–Gajic ('16-'21), Shlapentokh-Rothman–Teixeira da Costa ('20), Häfner–Hintz–Vasy ('19), Dafermos–Holzegel–Rodnianski ('19), Dafermos–Rodnianski–Shlapentokh-Rothman ('16), . . .

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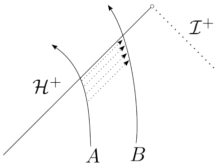

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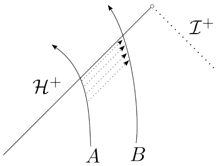

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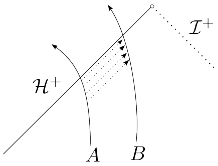

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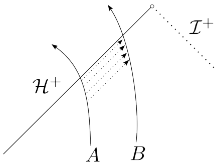

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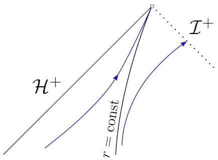

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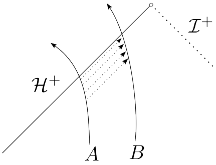

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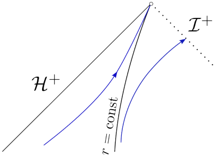

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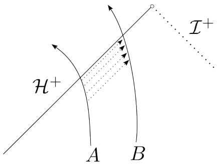

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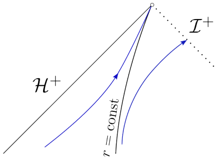

Precedent for studying the **massless Vlasov** equation to understand phenomena which are not well understood for other massless linear fields: Moschidis ('18,'20), Poisson–Israel ('89,'90)

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The Reissner–Nordström (**RN**) spacetime

Reissner–Nordström black hole of mass m and electrical charge $|q| \leq m$:

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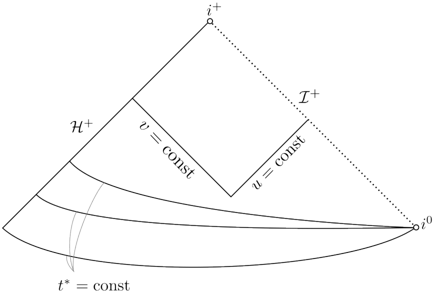

Figure: (t^*, r)-coordinates and double null coordinates

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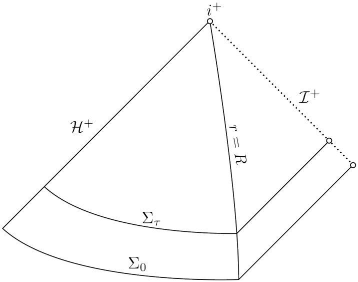

Figure: τ-time function

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We define moments of f for suitable polynomially-in-p bounded weights $w : \mathcal{P} \to \mathbb{R}$

$$\int_{\mathcal{P}_x} wf \, d\mu_x, \quad \text{e.g.}\, T^{\mu\nu}[f] = \int_{\mathcal{P}_x} p^\mu p^\nu f \, d\mu_x.$$

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Assume f solves the massless Vlasov equation on subextremal RN *and the initial distribution $f_0 : \mathcal{P}|_{\Sigma_0} \to [0, \infty)$ is smooth and compactly supported. Then for all $x \in M$ with $\tau(x) \geq 0$*

$$\int_{\mathcal{P}_x} wf \, d\mu_x \leq C \|f_0\|_{L^\infty} \frac{1}{r^2} e^{-c\tau(x)},$$

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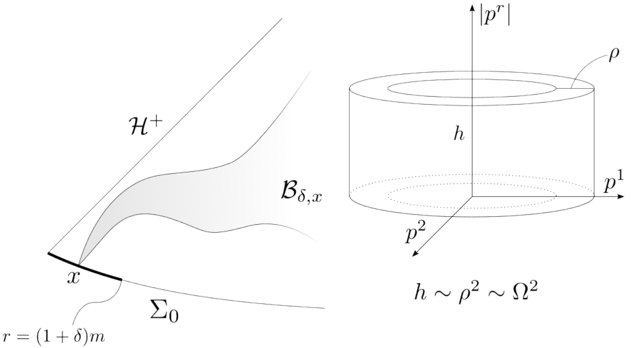

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Moreover the rate is sharp along the event horizon in general: for $x \in \mathcal{H}^+$

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We denote by T the timelike Killing derivative on **ERN**. We use (t^*, r)-coordinates:

Theorem 3 (Non-decay for transversal derivatives on extremal RN)

Assume that f solves the massless Vlasov equation on ERN *and f_0 is smooth and compactly supported. If we assume in addition that $Tf_0(x,p) \neq 0$ for $(x,p) \in \operatorname{supp}(f_0)$ and $\mathcal{B}_\delta \subset \operatorname{supp}(f_0)$ then for $x \in \mathcal{H}^+$ with $\tau(x) \gg 1$*

$$\left| \partial_r \int_{S^2} T^{t^* t^*}[f]\, d\omega \right| \geq C \left| \min_{(x,p)\in\mathcal{B}_\delta} |Tf_0(x,p)| \right|,$$

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where the horizon charge $H_0[\psi] = \frac{m^2}{4\pi} \int_{S^2} (\partial_{t^*} - \partial_r)(r\psi)|_{r=m} \, d\omega$ is conserved along $\mathcal{H}^+$.

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- Velozo (forthcoming): nonlinear stability of Schwarzschild as a solution to coupled spherically symmetric massless Einstein–Vlasov system

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Sketch of the proof: the subextremal case

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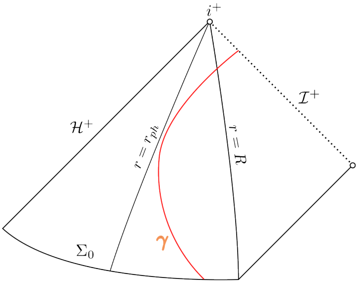

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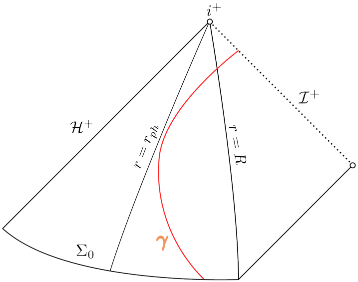

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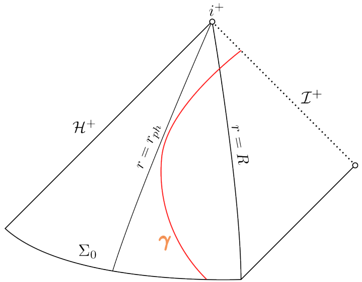

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Recall: geodesic flow on RN is completely integrable (Jacobi elliptic functions).

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Sketch of the proof: the subextremal case

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$$\tau(\gamma(s)) \lesssim 1 + |\log|\varepsilon|| + \mathfrak{s}\left|\log(1+|\varepsilon|)\Omega^2(r(0))\right|,$$

where $\mathfrak{s} = 1 + \mathrm{sgn}(p^r(0))$.

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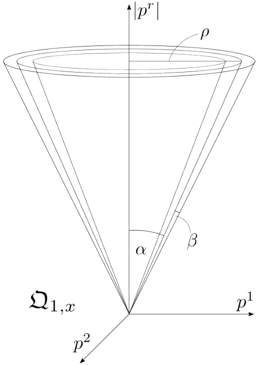

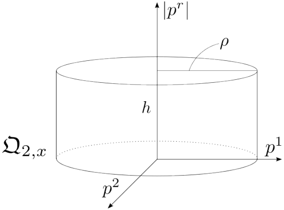

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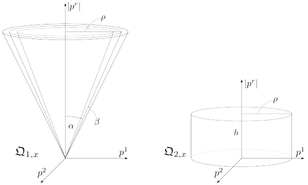

$$\int_{\mathcal{P}_x} wf\, d\mu_x \leq \left(\max_{\text{supp}\,(f)} |w|\right) \|f_0\|_{L^\infty} \left[\text{vol}(\mathfrak{Q}_{1,x}) + \text{vol}(\mathfrak{Q}_{2,x})\right]$$
$$\lesssim \|f_0\|_{L^\infty} e^{-c\tau(x)}.$$

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- Even though both p^r and $\not{p}$ are small as $\tau(x) \to \infty$:

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In fact: construct family of geodesics which cross $\mathcal{H}^+$ at arbitrarily late times while satisfying $p^{t^*} \sim 1$ on $\mathcal{H}^+ \rightsquigarrow$ allows to define $\mathcal{B}$.

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with $\mathcal{E} \lesssim \tau(x)^{-2}$.

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Therefore we find

$$\left| \int_{\mathfrak{Q}_{2,x}} \frac{(p^{t^*})^3}{|p^r|} \partial_{t^*} f \, d\mu_x \right| \gtrsim \tau(x)^2 \left(\min_{\mathcal{B}} |\partial_{t^*} f_0| \right) \mathrm{vol}(\mathfrak{Q}_{2,x})$$
$$\gtrsim \min_{\mathcal{B}} |\partial_{t^*} f_0| .$$

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